LOCAL BUCKLING ANALYSIS OF PULTRUDED FRP THIN-WALLED BEAMS AND COLUMNS

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Abstract
In this paper a mechanical model capable to predict the local buckling of pultruded FRP thin-walled beams and columns, taking into account the shear deformability of composite materials, is presented. The model is based on the individual analysis of buckling of the components of FRP profile, assumed as elastically restrained transversely isotropic plates. The analysis is developed within the field of small strains and moderate rotations.

Keywords: FRP, local buckling, thin-walled beams.

1. Introduction

In view of the advantages that composite materials exhibit in comparison with conventional ones, the use of pultruded Fiber Reinforced Polymers (FRP) thin walled beams represents an interesting challenge in the field of Civil Engineering.

Due to the thin-walled sectional geometry and relatively low stiffness of FRP materials, which are commonly made of glass fibers and polyester resins, problems associated with large elastic deformation and buckling are common in current design [1-6]. Moreover it is widely recognized that the mechanical response is strongly influenced by shear deformability, being shear elastic moduli sensitively small with respect to longitudinal ones. With particular reference to global buckling, it has been demonstrated that shear deformation can affect the ultimate failure of thin-walled FRP members [7-10].

In agreement with Italian guidelines for designing structures entirely made of composite materials [11], the local buckling analysis of thin walled FRP shapes can be accomplished by modeling individually the profile plate components (i.e. flanges and web in the case of open section), under the assumption of flexible plate junctions (i.e. flange-web junctions in the case of open section).
More specifically, each flat panel of FRP members can be modeled as a simply supported composite plate elastically restrained along the junctions. Within this context, some experimental [12] as well theoretical [13,14] approximated expression of the rotational restraint stiffness, depending on geometric and mechanical properties of profile components, are available in literature.

This work presents a mechanical model capable to predict the local buckling of pultruded FRP thin-walled beams and columns, taking into account the shear deformability of the composite.

The proposed model is developed starting from the buckling analysis of the single component of FRP profile, assumed as elastically restrained plates. With this aim, Mindlin-Reissner theory is developed in the field of small strains and moderate rotations [15], as well as it is extended to elastic transversely isotropic materials.

The numerical analysis here presented, performed by means of a weak formulation of the buckling problem within finite element method, examines the case of an “I” simply supported beam subject to an axial load.

The procedure is validated by a comparison with values numerically obtained by adopting the approach proposed in [16] for a beam under constant axial load.

2. Kinematics

Let consider an “I” beam with the constant cross-section of Figure 1. The adopted Cartesian reference frame has the origin in the centroid, G, of one of the bases of the beam, being the X and Y axes coincident with the central axes of inertia of the cross-section and the Z axis coincident with the longitudinal beam axis.

The cross section can be divided into five plate components, internally connected through flexible flange-web junctions.

\[ \text{Figure 1. Schematic representation of the pultruded profile cross section.} \]

Then, the single plate can be modeled as an elastically restrained transversely isotropic one.

Let us denote the undeformed mid-plane of the plate with the symbol \( \Omega_0 \). The total domain of the plate is \( \Omega_0 \times (-h/2, +h/2) \). The boundary of the total domain consists of surfaces \( S^+(z = +h/2) \), \( S^-(z = -h/2) \) and \( \Gamma = \partial \Omega_0 \times (-h/2, +h/2) \) (Figure 2). \( \Gamma \) is a curved surface, with outwards normal \( \hat{n} = n_x \hat{e}_x + n_y \hat{e}_y \), where \( n_x \) and \( n_y \) are the direction cosines of the unit normal.
The classical theory available for shear deformable plates, due to Mindlin and Reissner [17]-[18], is based on the following assumptions:
- straight lines perpendicular to the mid-plane (i.e. transverse normals) before deformation remain straight after deformations;
- the transverse normals do not experience elongation.

Therefore, the displacement field of the plate, referred to the reference system \{O, x, y, z\} (Figure 2), can be written in the following form:
\[ u = u(x, y, z) = u_0(x, y) + \Phi_y(x, y) \cdot z, \]  
\[ v = v(x, y, z) = v_0(x, y) - \Phi_x(x, y) \cdot z, \]  
\[ w = w(x, y) = w_0(x, y), \]  
where \((u_0, v_0, w_0)\) denote the displacements of a material point at the mid-plane of the plate and \((\Phi_x, \Phi_y)\) are the rotations of transverse normals about the \(x\)-axis and \(y\)-axis, respectively.

According to the hypothesis of small strains accompanied by moderate rotation and assuming that \(\omega_{xy}\) infinitesimal rotation tensor component is negligible, it is easy to show that the column vector, \(\mathbf{E}\), associated with Green–Saint Venant strain tensor assume the following expression:
\[ \mathbf{E} = \mathbf{e}^{(1)} + \mathbf{e}^{(2)} = \mathbf{e}^{(1c)} + \zeta \mathbf{e}^{(1)} + \mathbf{e}^{(2)}, \]  
\[ \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yx} \\ \sigma_{yy} \\ \sigma_{yz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\partial \Phi_y}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial \Phi_x}{\partial y} & 0 & 0 & 0 \\ \frac{\partial \Phi_y}{\partial x} & -\frac{\partial \Phi_x}{\partial y} & 0 & 0 & 0 & 0 \\ \frac{\partial \Phi_y}{\partial x} & \frac{\partial \Phi_x}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \zeta \begin{pmatrix} 1/8 (\Phi_y - \partial w_0/\partial x)^2 \\ 1/8 (\Phi_x + \partial w_0/\partial y)^2 \\ 1/4 (\Phi_x + \partial w_0/\partial y)(\Phi_y - \partial w_0/\partial x) \end{pmatrix}, \]  
\( (2a) \)
being \( \varepsilon^{(1c)} \) the first order constant strain vectors; \( z \cdot \varepsilon^{(1f)} \) the first order linear strain vector; \( \varepsilon^{(2)} \) the second order strain vector.

### 3. Buckling Analysis

In order to analyze buckling behavior of shear deformable elastic plates, the following displacement field, responsible for the transition from fundamental configuration to varied one, is considered:

\[
\Delta u(x, y, z) = \Delta \phi_x(x, y) \cdot z,
\]
\[
\Delta v(x, y, z) = -\Delta \phi_x(x, y) \cdot z,
\]
\[
\Delta w(x, y) = \Delta w_0(x, y).
\]  
(3a) (3b) (3c)

Thus, for small strains and moderate rotations, the strain-displacement relations take the form:

\[
\Delta \mathbf{E} = \Delta \varepsilon^{(1)} + \Delta \varepsilon^{(2)} + z \Delta \varepsilon^{(1c)} + \Delta \varepsilon^{(2)},
\]  
(4a)

\[
\begin{bmatrix}
\Delta E_{xx} \\
\Delta E_{yy} \\
\Delta E_{zz} \\
\Delta E_{xy}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & \frac{\partial \Delta \phi_x}{\partial x} \\
0 & 0 & -\frac{\partial \Delta \phi_x}{\partial y} \\
-\Delta \phi_x + \frac{\partial \Delta w_0}{\partial y} & 0 & 0 \\
\Delta \phi_x + \frac{\partial \Delta w_0}{\partial x} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
+ z \begin{bmatrix}
\frac{1}{8} \left( \Delta \phi_x - \frac{\partial \Delta w_0}{\partial x} \right)^2 \\
\frac{1}{8} \left( \Delta \phi_x + \frac{\partial \Delta w_0}{\partial y} \right)^2 \\
\frac{1}{8} \left( \Delta \phi_x + \frac{\partial \Delta w_0}{\partial y} \right)^2 \\
\frac{1}{4} \left( \Delta \phi_x - \frac{\partial \Delta w_0}{\partial x} \right) \left( \Delta \phi_x - \frac{\partial \Delta w_0}{\partial x} \right)
\end{bmatrix}.
\]  
(4b)

The fundamental configuration and the corresponding total potential energy are indicated by \( C^0 \) and \( E^0 \), respectively, whereas the symbols \( C^* \) and \( E^* \) indicate the varied configuration and the corresponding total potential energy, respectively.

The equilibrium conditions in the configurations \( C^0 \) and \( C^* \) can be expressed in a variational form by equating to zero the first variations of the corresponding total potential energy functionals \( E^0 \) and \( E^* \). In symbols:

\[
\delta E^0 = 0, \quad \delta E^* = 0.
\]  
(5a) (5b)

The Eqs. (5) can also be written as:

\[
\delta \left( E^* - E^0 \right) = \delta \Delta E = 0.
\]  
(6)

The hypothesis of small strains, available in the fundamental configuration, allow to express Eq. (6) in the form:

\[
\delta \left( U - V_2 + V_2^* \right) = 0.
\]  
(7)

The term \( U \) is the elastic energy corresponding to the transition from \( C^0 \) to \( C^* \) and the terms \( V_2 \) and \( V_2^* \) are, respectively, the second-order work carried out by the external forces and by the stresses acting in the fundamental configuration, when the plate configuration changes from \( C^0 \) to \( C^* \) [19].
On the basis of the results obtained in the previous paragraphs, the expression of the elastic energy for a plate with rotational restraints along the sides parallel to $x$ axis can be written as:

$$
U = \frac{1}{2} \int_{\Omega} \int_{b/2}^{h/2} A_j \Delta \varepsilon^{(i)} \Delta \varepsilon^{(j)} dzd\Omega + \frac{1}{2} \int_{-L/2}^{L/2} \left[ k^{-} \left( \Delta \varphi_x \big|_{y=-L/2} \right)^2 + k^{+} \left( \Delta \varphi_x \big|_{y=L/2} \right)^2 \right] dx = 
$$

$$
= \frac{1}{2} \int_{\Omega} \left[ \left( \frac{\partial \Delta \varphi_x}{\partial x} A_{44} + \frac{\partial \Delta \varphi_x}{\partial y} A_{45} \right) \frac{\partial \Delta \varphi_x}{\partial x} + \left( \frac{\partial \Delta \varphi_y}{\partial x} A_{45} + \frac{\partial \Delta \varphi_y}{\partial y} A_{55} \right) \frac{\partial \Delta \varphi_y}{\partial y} + \right. 
\left. \left( \frac{\partial \Delta \varphi_y}{\partial y} A_{45} - \frac{\partial \Delta \varphi_x}{\partial x} \right) A_{46} \right] + \left. \left[ -\Delta \varphi_x + \frac{\partial \Delta w_0}{\partial y} \right]^2 A_{77} \right. 
\left. + \left[ \Delta \varphi_y + \frac{\partial \Delta w_0}{\partial x} \right]^2 A_{88} \right] \right] d\Omega + 
\left. + \frac{1}{2} \int_{-L/2}^{L/2} \left[ k^{-} \left( \Delta \varphi_x \big|_{y=-L/2} \right)^2 + k^{+} \left( \Delta \varphi_x \big|_{y=L/2} \right)^2 \right] dx. 
$$

In Eqs. (8) $k^{-}$ and $k^{+}$ represent the rotational spring stiffnesses (Figure 3) and there are:

$$
A_{44} = \frac{h^3}{12} \left( \frac{E_x}{1-\nu_x^2} \right), A_{45} = \frac{h^3}{12} \left( \frac{v_{x y} E_y}{1-\nu_y^2} \right), A_{55} = \frac{h^3}{12} \left( \frac{E_y}{1-\nu_y^2} \right), 
$$

$$
A_{66} = \frac{h^3}{12} G_{x y}, A_{77} = \chi_x h G_{x y}, A_{88} = \chi_y h G_{x y}, 
$$

being $E_x, E_y, v_{x y}, G_{x y}, G_{x y}$ the independent elastic parameters, $\chi_x$ and $\chi_y$ the shear factors in $x$ and $y$ direction, respectively.

Since the applied loads do not work for the second order displacement responsible for the transition from configuration $C^0$ to $C^*$, the second-order work carried out by external forces, $V_2$, is zero.

The second-order work, $V_2^*$, carried out by stresses in the fundamental equilibrium configuration can be written as follows:

$$
V_2^* = \lambda \int_{\Omega} \int_{-h/2}^{h/2} \sigma_i^{0} \Delta \varepsilon^{(2)} dz d\Omega = 
$$

$$
= \frac{1}{4} \lambda \int_{\Omega} \left[ \frac{1}{2} N_x^0 \left[ (\Delta \varepsilon_x)^2 + \frac{\partial \Delta w_0}{\partial x} - 2 \Delta \varphi_x \frac{\partial \Delta w_0}{\partial x} \right] + 
\frac{1}{2} N_y^0 \left[ (\Delta \varepsilon_y)^2 + \frac{\partial \Delta w_0}{\partial y} - 2 \Delta \varphi_y \frac{\partial \Delta w_0}{\partial y} \right] + 
- N_{x y}^0 \left[ \Delta \varepsilon_x \Delta \varphi_x - \Delta \varphi_x \frac{\partial \Delta w_0}{\partial x} + \frac{\partial \Delta w_0}{\partial y} \Delta \varphi_y - \frac{\partial \Delta w_0}{\partial y} \frac{\partial \Delta w_0}{\partial x} \right] \right] d\Omega, 
$$

where $\lambda$ is a constant by which the external loads must be multiplied to cause buckling, $N_x^0, N_y^0, N_{x y}^0$ are the normal forces per unit length in the fundamental equilibrium configuration, derived starting from $\sigma_i^{0}$ components of the column vector associated with the Piola-Kirchhoff tensor.
4. Finite element discretization

In order to apply the finite element method the mid-plane of the plate is subdivided in rectangular elements (Figure 3).

Assuming a quadratic 9-node Lagrange finite element, the above introduced displacements are independently interpolated as follows:

\[ \Delta w_0^{(e)} = \sum_{i=1}^{9} \psi_i(\xi, \eta) \Delta w_{0i}^{(e)} , \]  
\[ \Delta \Phi_x^{(e)} = \sum_{i=1}^{9} \psi_i(\xi, \eta) \Delta \Phi_{xi}^{(e)} , \]  
\[ \Delta \Phi_y^{(e)} = \sum_{i=1}^{9} \psi_i(\xi, \eta) \Delta \Phi_{yi}^{(e)} , \]

where \( \psi_i(\xi, \eta) \) are the nine lagrangian quadratic shape functions [20]; \( \Delta w_{0i}^{(e)} \), \( \Delta \Phi_{xi}^{(e)} \) and \( \Delta \Phi_{yi}^{(e)} \) represent the displacement components of the i-th node of the e-th finite element.

Starting from the FEM approach, the proposed model has been validate by comparing the theoretical prediction of buckling of an elastically restrained isotropic plate with the results reported in [16].

5. The case of a simply supported beam subject to axial load

Let us consider an “I” simply supported beam subject to axial load, \( F \). The normal forces per unit length in the fundamental equilibrium configuration are given by the following relationships:

\[ N_x^0(x, y) = \frac{F}{A_{FRP}} t_p, \quad N_y^0(x, y) = N_{xy}^0(x, y) = 0, \]  

where \( A_{FRP} \) is the area of the profile cross-section and \( t_p \) is the thickness of FRP plate component. The boundary conditions for the web can be expressed as follows:

\[ \Delta w_0(x, y = \pm L_s / 2) = \Delta w_0(x = \pm L_s / 2, y) = 0, \]  
\[ \Delta \Phi_x(x = \pm L_s / 2, y) = 0, \]
Δφ_y (x, y = ±L_y / 2) = 0, \quad (12c)

k^- = k^+ = k \quad (12d)

On the other hand, the boundary conditions for the half flange can be expressed as follows:

Δw_0 (x, y = L_y / 2) = Δw_0 (x = ±L_x / 2, y) = 0, \quad (14a)

Δφ_x (x = ±L_x / 2, y) = 0, \quad (13b)

Δφ_y (x, y = −L_y / 2) = 0, \quad (13c)

k^- = k, \quad k^+ = 0. \quad (13d)

Assuming F = 1 N, a parametric study is performed in order to investigate the influence of the rotational spring stiffnesses and the length-width ratio on the buckling load of some pultruded GFRP (Glass Fiber Reinforced Polymer) “I” simply supported beams, characterized by the geometrical and mechanical properties listed in Table 1.

### Table 1 – Dimensions and mechanical properties of the beams.

<table>
<thead>
<tr>
<th>w_f</th>
<th>w_w</th>
<th>t_f</th>
<th>t_w</th>
<th>E_x</th>
<th>E_y</th>
<th>v_xy</th>
<th>G_xy</th>
<th>G_yx</th>
</tr>
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<td>[mm]</td>
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<td>[mm]</td>
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<tr>
<td>100.00</td>
<td>100.00</td>
<td>10.00</td>
<td>10.00</td>
<td>30.00</td>
<td>10.00</td>
<td>0.25</td>
<td>3.00÷6.00</td>
<td>4.50</td>
</tr>
</tbody>
</table>

Let us introduce the buckling coefficients K_{b,w} and K_{b,f}, as well as the adimensional quantities R_w and R_f:

K_{b,w} = \frac{λ \cdot F \cdot t_w \cdot w_w^2}{π^2 \cdot A_{FRP} \cdot A_{44}}; \quad K_{b,f} = \frac{λ \cdot F \cdot t_f \cdot (w_f / 2)^2}{π^2 \cdot A_{FRP} \cdot A_{44}} \quad (15a)

R_w = \frac{k}{E_x \cdot t_w^2}; \quad R_f = \frac{k}{E_x \cdot t_f^2} \quad (16b)

The values of K_{b,w} versus the length-width ratio for multiple values of R_w are depicted in Figure 4, while the values of K_{b,f} versus the length-width ratio for multiple values of R_f are reported in Figure 5.
In order to highlight the influence of shear stiffness on local bucking, further flange buckling curves versus length-width ratio are depicted, with reference to different values of shear modulus of elasticity, $G_{xy}$. 

Figure 4. Web buckling curves – $G_{xy}$=3.00 GPa.

Figure 5. Flange buckling curves - $G_{xy}$=3.00 GPa.
6. Conclusions

A mechanical model capable to take into account the shear deformability of the composite material on the local buckling behavior of pultruded FRP thin-walled beams and columns has been proposed.

The local buckling of an “I” simply supported beam subject to axial load has been investigated by means of a finite element approximation, assuming several values of $G_{xy}$ shear modulus of elasticity.

The results, referred to profiles characterized by $w_f = w_w$, have underlined that local buckling occurs in the flanges, due to the presence of supports only on one side of the plate, as well as the flange buckling coefficient variation tends to be negligible for values of length-width ratio greater than 5.

Moreover, the influence of rotational springs stiffnesses on the buckling of flange appears relevant for values of length-width ratio greater than 2.

Finally, the analysis has highlighted that the effect of shear stiffness on the buckling load of the FRP profiles can be consistent.

References


