THEORETICAL ANALYSIS OF INTERFACIAL STRESSES IN CURVED MEMBERS BONDED WITH A THIN PLATE

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ABSTRACT

The use of externally bonded fibre-reinforced polymer (FRP) laminates to strengthen concrete, masonry, timber or metallic structures is a technique that has become very popular. The effectiveness of this technique hinges heavily on the performance of the bond between the FRP laminate and the substrate, which has been the subject of many existing studies. In particular, the elastic interfacial stresses between a beam and a soffit plate have been the subject of numerous analytical studies. Surprisingly, none of these studies has examined interfacial stresses in members with a curved soffit, despite that such members are often found in practice. This paper presents an analytical study to examine the interfacial stresses between a curved member of uniform section size and a thin plate bonded to its soffit. The governing differential equations for the interfacial shear and normal stresses are formulated and then solved with appropriate simplifying assumptions. A numerical example is presented to illustrate the effect of the curvature of the member on the interfacial stress distributions. The analytical solution is verified by comparing its predictions with those from a finite element model.

KEYWORDS

Curved beams, curved members, FRP, interfacial stresses, plate bonding, strengthening.

INTRODUCTION

One of the simplest methods to retrofit an existing beam is to bond a fibre-reinforced polymer (FRP) or steel plate to its soffit. This technique has been widely used to retrofit reinforced concrete (RC) beams and beams of other materials such as wood, iron or steel. In such plated beams, debonding of the soffit plate from the beam is an undesirable failure mode as it is a brittle mode and prevents the full utilization of the tensile strength of the plate material. It is thus important to be able to predict the debonding failure load. Debonding failures starting from a plate end depend largely on the concentration of interfacial shear and normal stresses between the beam and the bonded plate in the vicinity of the plate end. The determination of these interfacial stresses in the elastic range has thus been extensively researched. In particular, several relatively simple approximate closed-form solutions for interfacial stresses have been developed. Smith and Teng (2001) presented a review of some of these solutions. Surprisingly, none of the existing studies has examined interfacial stresses in members with a curved soffit. Nevertheless, such members are often found in practice. Classical examples are arch bridges, and metallic, concrete or timber curved beams used to cover large spans. This paper presents a simple, approximate analytical solution for the interfacial stresses between a curved member of uniform section size and a thin plate bonded to its soffit. This closed-form solution provides a useful but simple tool for understanding the interfacial behaviour and for exploitation in developing design methods. While the governing differential equations presented in this paper cater for both load cases of a point load (Figure 1a) and a uniformly distributed vertical load (Figure 1b), a complete solution is only presented for the former case, due to space limitations. A more detailed derivation of the solution is given in De Lorenzis et al. (2005), where a solution for the latter case is also presented.

ASSUMPTIONS OF THE MODEL

Figure 2 shows a differential element of a plated curved beam. The position coordinate can be the angle θ or the curvilinear abscissa s (Figure 1). If r is the radius of curvature of the mid-adhesive axis and s is measured along the same axis, the relationship between the two is

\[ s = r \theta \]  

(1)
with $\theta$ measured in radians. In this paper, it is assumed for simplicity that $r$ is constant along the span of the beam, and that it is much larger than the cross-sectional dimensions. Both assumptions are normally valid for civil engineering structures, and considerably simplify the derivation and solution of the governing differential equations for normal and shear stresses at the interface between the beam and the strengthening plate. In Figure 2, the interfacial shear and normal stresses are denoted by $\tau(\theta)$ and $\sigma(\theta)$, respectively (they can be equivalently expressed as $\tau(s)$ and $\sigma(s)$). The figure also shows the positive directions for the bending moment, shear force, axial force and applied load. The distributed load $q$ acts in the vertical direction, and hence is inclined at a variable angle to the beam axis (see also Figure 1b). All three materials are assumed to be linearly elastic.

**EQUILIBRIUM EQUATIONS**

In the following, $y_1$ and $y_2$ are the distances from the bottom of adherend 1 (the beam) and the top of adherend 2 (the plate) to their respective centroids; $y_g$ is the distance of the centroid of the plated cross-section from the mid-adhesive axis; $b_2$ is the width of the strengthening plate. Note that the resultants of the interfacial shear and
normal stresses $\tau(0)$ and $\sigma(0)$ are taken equal to $(\tau_b, r \theta_0)$ and $(\sigma_b, r \theta_0)$, respectively, neglecting the fact that the lengths of top and bottom circumferences of the adhesive layer differ by $(t_a, d \theta_0)$. This assumption is justified by the adhesive thickness being very small compared to the radius of curvature $r$. For the same reason, $t_a$ is considered negligible in the following equations. Table 1 lists the equilibrium equations for the differential elements of adherends 1 and 2 and of the plated beam. Note that the equilibrium equations for the element of plated beam can be obtained from the corresponding equations for elements 1 and 2, being $N = N_1 - N_2$, $V = V_1 + V_2$, and $M = M_1 + M_2 + N_1(y_1 - y_2) + N_2(y_2 + y_3)$.

<table>
<thead>
<tr>
<th>Table 1. Equilibrium equations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equilibrium of forces in the tangential direction</strong></td>
</tr>
<tr>
<td>for element 1 (2a)</td>
</tr>
<tr>
<td>$dN_1 = \tau_b - \frac{V_1}{r} - q(1 + \frac{2y_1}{r}) \sin \left( \frac{\alpha_1 + s}{r} \right) \cos \left( \frac{\alpha_1 + s}{r} \right)$</td>
</tr>
<tr>
<td>$dV_1 = \frac{N_1}{r} - q(1 + \frac{2y_1}{r}) \sin \left( \frac{\alpha_1 + s}{r} \right) - \sigma b_1$</td>
</tr>
<tr>
<td>$dM_1 = V_1 + \frac{N_1}{r} - q(1 + \frac{2y_1}{r}) \sin \left( \frac{\alpha_1 + s}{r} \right) \cos \left( \frac{\alpha_1 + s}{r} \right)$</td>
</tr>
</tbody>
</table>

**COMPATIBILITY EQUATIONS**

In order to simplify the derivation of the differential equations and to arrive at a closed-form solution, the compatibility equations are written herein neglecting the effect of curvature. In other words, the solution is exact with respect to equilibrium but approximate with respect to compatibility. The error so introduced is expected to be acceptably small if the radius of curvature is large compared with the cross-sectional dimensions of the beam. This assumption is usually valid for civil engineering structures.

The general expression for the shear strain in polar coordinates is

$$\gamma_{\theta \theta} = \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\theta}{\partial r}$$

(6)

It is now assumed that the second and the third terms in the right hand side of Eq. 6 are negligible with respect to the first one. This assumption is expected to lead to little error if $r$ is large compared to the cross-sectional dimensions of the beam. It is further assumed that the shear strain in the adhesive layer is uniform across the adhesive thickness, which also means that the shear stress $\tau(s)$ is uniform across the adhesive thickness. Hence,

$$\tau(s) = \frac{G_a}{t_s} \left[ u_{20}(s) - u_{10}(s) \right]$$

(7)

where $u_{20}$ is the displacement along the $\theta$ axis of the top fibre of adherend 2, $u_{10}$ is the displacement along the $\theta$ axis of the bottom fibre of adherend 1, and $G_a$ is the shear modulus of the adhesive. The general expression for the normal strain $\varepsilon_r$ in polar coordinates is:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

(8)

It is assumed herein that $\varepsilon_r$ is constant across the adhesive thickness, i.e. the normal stress $\sigma(s)$ is constant across the adhesive thickness. Hence

$$\sigma(s) = \frac{E_a}{t_s} \left[ u_{2r}(s) - u_{1r}(s) \right]$$

(9)

where $u_{2r}$ is the displacement along the $r$ axis of the top fibre of adherend 2, $u_{1r}$ is the displacement along the $r$ axis of the bottom fibre of adherend 1, and $E_a$ is the Young’s modulus of the adhesive.

**INTERFACIAL SHEAR STRESS: GOVERNING DIFFERENTIAL EQUATION**

The first derivative of the interfacial shear stress (Eq. 7) is as follows:
Assuming that in the following general expression for $\varepsilon_\theta$ in polar coordinates:

$$
\varepsilon_\theta(s) \approx \frac{du}{ds} - \frac{1}{E_1 A_1} N_1(s)
$$

where $A_1$ and $A_2$ are the cross-sectional areas of adherends 1 and 2, respectively, $I_1$ and $I_2$ are their respective second moments of area, and $E_1$ and $E_2$ are their respective elastic moduli. Substituting Eqs 12a and 12b into Eq. 10 yields

$$
\frac{d\tau}{ds} = \frac{G_a}{t_\sigma} \left[ \frac{du_{10}}{ds} - \frac{du_{20}}{ds} \right]
$$

Assuming that the beam and the soffit plate have the same curvature after deformation, the relationship between the moments in the two adherends can be expressed as

$$
M_2(s) = R M_1(s) \quad \text{with} \quad R = \frac{E_1 I_1}{E_2 I_2}
$$

Note that Eq. 14 is an approximation which was also used by previous researchers studying the bond behaviour between a straight beam and a strengthening plate, and was found to lead accurate results (Smith and Teng 2001). $M_2$ can be expressed from Eq. 14 as a function of $M_1$, and $N_2$ can be expressed from Eq. 5a as a function of $N_1$. Substitution of both expressions into Eq. 5c yields

$$
\frac{d^2 \tau}{ds^2} = 1 + R_\alpha
$$

By defining the following non-dimensional coefficients:

$$
c_1 = \frac{E_1 A_1 \left( y_1 + y_2 \right)^2}{E_1 I_1} \quad \text{with} \quad c_2 = 1 + R_\alpha
$$

Eq. 19 after differentiation with respect to $s$ becomes

$$
\frac{dM_1}{ds} = \frac{\left( y_1 + y_2 \right)}{y_1 + y_2 - r} \left[ \frac{\alpha_\theta + s}{r} \right] \cos \left( \frac{\alpha_\theta + s}{r} \right)
$$

Eliminating $V_1$ from Eqs 2a and 4a, the following equation is obtained:

$$
\frac{dM_1}{ds} = \left( y_1 + y_2 \right) \left( y_1 + r \right) \frac{dN_1}{ds} - \frac{a}{r} \left( r + 2y_1 \right)^2 \sin \left( \frac{\alpha_\theta + s}{r} \right) \cos \left( \frac{\alpha_\theta + s}{r} \right)
$$

which, combined with the first derivative of $N_1$ given by Eq. 16, yields

$$
\frac{dM_1}{ds} = -d_1 b_1 \tau + d_2 \frac{dM}{ds} + \left( y_2 + y_\theta \right) b_2 \frac{dN_1}{ds} \left( y_2 + y_\theta \right) b_2 \frac{dN_1}{ds} + d_1 \frac{a}{r} \left( r + 2y_2 \right)^2 \sin \left( \frac{\alpha_\theta + s}{r} \right) \cos \left( \frac{\alpha_\theta + s}{r} \right)
$$

where $d_1$ and $d_2$ are non-dimensional coefficients given by

$$
d_1 = \frac{\left( y_1 + y_2 \right)}{y_1 + r - \left( y_2 - r \right)} \quad \text{and} \quad d_2 = \frac{\left( y_1 + r \right) R}{y_1 + r - \left( y_2 - r \right) R}
$$

Finally, Eqs 21 and 23 can be combined into the following second order differential equation in the unknown function $\tau(s)$:
It is straightforward to demonstrate that, in the case of a straight beam of uniform section size \((r \to \infty)\), Eq. 25 reduces to the same equation derived by Smith and Teng (2001) for straight beams.

**INTERFACIAL NORMAL STRESS: GOVERNING DIFFERENTIAL EQUATION**

The second derivative of the normal stress given by Eq. 9 is as follows:

\[
\frac{d^2 \sigma}{ds^2} = \frac{E_a}{t_a} \left[ \frac{d^2 u_{2r}}{ds^2} - \frac{d^2 u_{2\theta}}{ds^2} \right]
\]

(26)

For both adherends 1 and 2 in bending,

\[
\frac{d^2 u_{2r}}{ds^2} = -\frac{M_2}{E_2 l_2} \quad \frac{d^2 u_{2r}}{ds^2} = -\frac{M_1}{E_1 l_1}
\]

(27a-b)

Eq. 26 thus becomes

\[
\frac{d^2 \sigma}{ds^2} = \frac{E_a}{t_a} \left[ \frac{M_2}{E_2 l_2} + \frac{M_1}{E_1 l_1} \right]
\]

(28)

Eliminating \(V_2\) from Eqs 2b and 4b, the following equation is obtained:

\[
\frac{d^2 \sigma}{ds^2} = -\frac{q d_1}{2 r E_1 l_1} \left( r + 2 y_1 \right)^2 \sin \left[ 2 \left( \alpha_a + \frac{s}{r} \right) \right]
\]

(30)

Substituting Eqs 29 and 22 into Eq. 28 after differentiation once more and using Eq. 5a yield

\[
\frac{dN_1}{ds} = \frac{R y_1 + r - (y_2 - r)}{y_1 + y_2} \frac{dN_2}{ds}
\]

(24c)

Substituting Eq. 2b, after differentiation with respect to \(s\), into Eq. 3b and using Eq. 5a, the following differential equation is obtained for \(N_1\):

\[
r \frac{d^2 N_1}{ds^2} + \frac{1}{r} \frac{dN_1}{ds} - b_2 r \frac{dr}{ds} - b_2 \sigma = \frac{N}{r} + r \frac{d^2 N}{ds^2}
\]

(32)

Differentiation of Eq. 32 with respect to \(s\) gives

\[
r \frac{d^3 N_1}{ds^3} + \frac{1}{r} \frac{dN_1}{ds} - b_2 r \frac{d^2 r}{ds^2} - b_2 \frac{d\sigma}{ds} = \frac{1}{r} \frac{dN}{ds} + r \frac{d^3 N}{ds^3}
\]

(33)

Substituting Eq. 31 into Eq. 33 and dividing the resulting equation by \(r\) yield the following fifth order differential equation for \(\sigma(s)\):

\[
\frac{d_1}{y_1 + y_2} \frac{R + 1}{E_2 l_2} \frac{d^2 \sigma}{ds^2} + \frac{d_1}{y_1 + y_2} \frac{R + 1}{E_1 l_1} \frac{d^2 \sigma}{ds^2} + \frac{b_2}{R} \frac{d\sigma}{ds} = \left[ \frac{d_1}{y_1 + y_2} \frac{R + 1}{E_2 l_2} \right] \frac{d^2 \tau}{ds^2} + \left[ \frac{d_1}{y_1 + y_2} \frac{R + 1}{E_1 l_1} \right] \frac{d^2 \tau}{ds^2} + 3 \frac{d_1}{E_1 l_1} \frac{dN}{ds} \left( d_3 + 1 \right) \frac{d^3 N}{ds^3} + \frac{3}{2} \frac{R d_1}{E_1 l_1} \left( r + 2 y_1 \right)^2 \sin \left[ 2 \left( \alpha_a + \frac{s}{r} \right) \right]
\]

(34)

\[\text{SIMPLY-SUPPORTED CURVED BEAM UNDER A POINT LOAD: INTERFACIAL SHEAR STRESSES}\]

**Differential Equation and Assumed Geometry**

Eq. 25 can be expressed as follows:

\[
\frac{d^2 \tau}{ds^2} + \lambda^2 \tau + m_1 \frac{dM}{ds} + m_2 \frac{dN}{ds} = 0
\]

(35)

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where \( \lambda = \sqrt{\frac{c_1 d_1 b_2 r G_s}{E_1 A_1 t_s (y_1 + y_2)}} \)  
\( m_1 = \frac{G_s}{E_1 A_1 t_s (y_1 + y_2)} (c_1 d_2 - c_2) \)  
\( m_2 = \frac{G_s c_1 d_2}{E_1 A_1 t_s} \frac{R_s}{1 + R_s} \)  
(36a-c)

It is assumed that the beam is supported with a hinge at one end and a roller at the other end, and is loaded with a point load \( F \). Symbols adopted in the following are defined in Figure 1a. It is also assumed that the load is situated on the right of the left termination of the strengthening plate, i.e. that \( \Delta x_a < \Delta x_b \).

The beam span, measured between the extreme points of the centroidal axis of the unplated beam, is related to the angle \( \alpha_0 \) by the following equation:

\[
1 = 2 \left( r + \frac{t_a}{2} + y_1 \right) \cos \alpha_0 \geq 2(r + y_1) \cos \alpha_0 
\]

(37)

Figure 1 also leads to

\[
\Delta x_a = r (\cos \alpha_0 - \cos \alpha_a) + y_1 \cos \alpha_0 
\]

\[
\Delta x_b = r (\cos \alpha_0 - \cos \alpha_b) + y_1 \cos \alpha_0 
\]

(38a-b)

The reactions \( R_A \) and \( R_B \) can be immediately found from equilibrium to be:

\[
R_A = \frac{F (r + y_1) \cos \alpha_0 + r \cos \alpha_b}{2 (r + y_1) \cos \alpha_0} 
\]

\[
R_B = \frac{F (r + y_1) \cos \alpha_0 - r \cos \alpha_a}{2 (r + y_1) \cos \alpha_0} 
\]

(39a-b)

General solution of the differential equation and boundary conditions

The solution of the differential equation must be found separately for the two portions of the plate to the left and to the right of the point load. For example, for the portion of the plate to the left of the point load, i.e. for

\[
0 \leq \theta \leq \alpha_a - \alpha, \quad \theta = \alpha_a - \theta, \quad a \leq \theta \leq 0,
\]

the general solution of Eq. 35 is

\[
\tau_1(s) = C_{1F} e^{\lambda_1 s} + C_{2F} e^{-\lambda_1 s} + \tau_{1F}(s)
\]

(40)

where \( \tau_{1F}(s) \) is a particular solution of Eq. 35. The internal axial force and bending moment are as follows:

\[
N(\theta) = R_A \cos(\alpha_a + \theta) 
\]

\[
M(\theta) = R_A \left[ (r + y_1) \cos(\alpha_0 - \theta) - (r + y_2) \cos(\alpha_a + \theta) \right]
\]

(41a-b)

from which the first derivatives expressed in terms of \( s \) are found to be:

\[
\frac{dN}{ds} = \frac{R_A}{r} \sin(\alpha_a + \frac{s}{r}) 
\]

\[
\frac{dM}{ds} = R_A \left( 1 + \frac{y_2}{r} \right) \sin(\alpha_a + \frac{s}{r}) 
\]

(42a-b)

A particular solution can be assumed to be of the following form:

\[
\tau_{1F}(s) = A_F \sin\left( \alpha_a + \frac{s}{r} \right)
\]

(43)

If Eqs 42 and 43 are substituted into Eq. 35, the following expression for the unknown constant \( A_F \) is obtained:

\[
A_F = R_A \frac{1}{\lambda^2 + \frac{1}{r^2}} 
\]

(44)

An analogous general solution with two unknown constants \( C_{1F} \) and \( C_{2F} \) can be found for the portion of the plate to the right of the point load (for more details, see De Lorenzis et al. 2005). Constants \( C_{1F}, C_{2F}, C_{3F} \) and \( C_{4F} \) must be found by applying four boundary conditions. At \( s = 0 \) (left end of the plate), \( M_2 = 0 \) and \( N_2 = 0 \), as a result, Eqs 5 yield:

\[
N = N_1 
\]

\[
M = M_1 + N_1 (y_1 - y_2)
\]

(45a-b)

Using Eq. 45, the first boundary condition can be obtained from Eq. 13:

\[
\frac{d\tau}{ds}_{s=0} = \frac{G_s}{t_s} \left[ - \frac{y_1}{E_1 I_1} M(0) + \frac{y_1}{E_1 I_1} N(0) (y_1 - y_2) + \frac{1}{E_1 A_1} N(0) \right]
\]

(46)

with \( M(0) \) and \( N(0) \) computed from Eqs 41. Similar boundary conditions can be written for \( s = r (\pi - 2\alpha_a) \). Here, \( M_2 = 0 \) and \( N_2 = 0 \), and Eqs 45 are still valid. Two more boundary conditions can be written under the point load, i.e. at \( s = r (\alpha_b - \alpha_a) \), imposing the continuity of the shear stress and that of its first derivative.

SIMPLY-SUPPORTED CURVED BEAM UNDER A POINT LOAD: INTERFACIAL NORMAL STRESSES

Differential Equation

The differential equation (Eq. 34) can be rewritten as follows:
\[
\frac{d^3 \sigma}{ds^3} + a_2 \frac{d^2 \sigma}{ds^2} + a_3 \frac{d \sigma}{ds} + a_4 \frac{d^2 \tau}{ds^2} + a_5 \tau + a_6 \frac{dN}{ds} + a_7 \frac{d^3 N}{ds^3} = 0
\]  
(47)

where constants \(a_2\) to \(a_7\) are given by:

\[
a_2 = \frac{1}{r^2} \quad \quad \quad a_3 = \frac{y_1 + y_2}{d_1} \quad \quad \quad a_4 = \left( \frac{y_1}{E_1 I_1} - \frac{y_2}{E_2 I_2} \right) \frac{b_2 E_4}{a_7} \quad \quad \quad a_5 = \frac{R + 1}{R} \frac{b_2 E_4}{a_7} \frac{(d_3 + 1)}{d_1} \quad \quad \quad a_7 = \frac{d_3 + 1}{d_1} \frac{y_1 + y_2}{d_1} \frac{E_4}{a_7} \quad \quad \quad a_7 = \frac{d_3 + 1}{d_1} \frac{y_1 + y_2}{d_1} \frac{E_4}{a_7}
\]  
(48a-c)

General Solution of the Differential Equation and Boundary Conditions

The general solution of the homogeneous differential equation corresponding to Eq. 47 is given by:

\[
\sigma_{sf}(s) = c^{\alpha s} (B_{1f} \cos \beta s + B_{2f} \sin \beta s) + c^{-\alpha s} (B_{1f} \cos \beta s + B_{2f} \sin \beta s) + B_{3f}
\]  
(49)

where \(\alpha = \sqrt{\frac{2(a_1 - a_2)}{4}}\) \(\beta = \sqrt{\frac{2(a_3 + a_2)}{4}}\)  
(50a-b)

It is assumed that for large values of \(s\) the general solution of the homogeneous equation does not diverge, which requires that \(B_1 = B_2 = 0\), so that Eq. 49 simplifies to:

\[
\sigma_{sf}(s) = c^{\alpha s} (B_{1f} \cos \beta s + B_{2f} \sin \beta s) + B_{3f}
\]  
(51)

The particular solution of Eq. 47 can be assumed to be of the following form:

\[
\sigma_{pf}(s) = C_F e^{\lambda s} - D_F e^{-\lambda s} + E_F \cos \left( \alpha_F + \frac{s}{r} \right)
\]  
(52)

and the constants are given by:

\[
C_F = -C_{1f} \frac{d_2 \lambda^2}{\lambda^2 + a_2 \lambda + a_3 \lambda} \quad \quad \quad D_F = -C_{2f} \frac{d_2 \lambda^2}{\lambda^2 + a_2 \lambda + a_3 \lambda} \quad \quad \quad E_F = -A_F
\]  
(53a-c)

Hence the general solution of the given differential equation, in the proximity of the plate end, is:

\[
\sigma_F(s) = c^{\alpha s} (B_{1f} \cos \beta s + B_{2f} \sin \beta s) + B_{3f} + C_F e^{\lambda s} - D_F e^{-\lambda s} + E_F \cos \left( \alpha_F + \frac{s}{r} \right)
\]  
(54)

Three boundary conditions at the plate end must be used to compute the three unknown constants \(B_3, B_4, B_5\). At \(s = 0\) (left end of the plate), these conditions are \(M_2 = 0\) and \(N_2 = 0\), which yield Eqs 45a-b, and also \(V_2 = 0\), which, together with Eq. 5b, yields:

\[
V = V_1
\]  
(55)

The first boundary condition can be obtained from Eq. 28 which, combined with Eqs 45, becomes:

\[
\frac{d^2 \sigma}{ds^2} \bigg|_{s=0} = \frac{E_a}{E_1 I_1} \left[ M(0) - N(0)(y_1 - y_6) \right]
\]  
(56)

The second boundary condition can be obtained from Eq. 28 after differentiation which, combined with Eqs 45, becomes:

\[
\frac{d^2 \sigma}{ds^2} \bigg|_{s=0} = \frac{t_a}{E_a} \frac{d^2 \sigma}{ds^2} \bigg|_{s=0} = \frac{E_a}{E_1 I_1} \left[ M(0) - N(0)(y_1 - y_6) \right] V(0)
\]  
(57a)

The third boundary condition can be obtained as follows. Eq. 32 written for \(s = 0\) yields:

\[
\sigma(0) = \frac{r}{b_2} \frac{d^2 N_1}{ds^2} \bigg|_{s=0} - \frac{r}{b_2} \frac{d^2 N_1}{ds^2} \bigg|_{s=0} - \tau \frac{d^2 N_1}{ds^2} \bigg|_{s=0} - \tau \frac{d^2 N_1}{ds^2} \bigg|_{s=0}
\]  
(57b)

Eq. 57a, combined with Eq. 31 after differentiation, yields the third boundary condition (in general for \(q \neq 0\)):

\[
\sigma(0) = \frac{t_a E_1 I_1}{y_1 + y_2} \frac{d^2 \sigma}{ds^2} \bigg|_{s=0} + \left( \frac{R + 1}{R} \frac{d \tau}{y_1 + y_2} \right) \frac{d^2 \sigma}{ds^2} \bigg|_{s=0} - \frac{r}{b_2} \frac{d^2 N_1}{ds^2} \bigg|_{s=0} - \frac{q d_1 (r + 2 y_1)^2}{b_2 R (y_1 + y_2)} \cos 2 \alpha_F
\]  
(58)

NUMERICAL EXAMPLE AND COMPARISON WITH FINITE ELEMENT RESULTS

The example considered here is a reinforced concrete beam with \(H_1 = 300\) mm, \(b_1 = 200\) mm, and \(E_1 = 30\) GPa, bonded with a thin carbon FRP (CFRP) plate having \(H_2 = 1.2\) mm, \(b_2 = 200\) mm, and \(E_2 = 165\) GPa. The
adhesive is assumed to have a thickness $t_a = 2$ mm, an elastic modulus $E_a = 4$ GPa, and a Poisson’s ratio $\nu_a = 0.35$. The span of the beam $l = 3$ m, the curvilinear distance from the termination of the plate to the end of the beam soffit is $300$ mm, the point load $F = 150$ kN and is located at mid-span. The radius of curvature $r$ is made variable between $r_{\text{min}} = 1.35$ m, corresponding to a semicircular beam, and $r_{\text{max}} = \infty$, corresponding to a beam with a flat soffit.

Figures 3 and 4 show the interfacial shear and normal (peeling) stresses near the plate end as solid lines. It is evident that, except for the semicircular beam where both stresses at the plate end are close to zero, the peak values of both stresses are predicted to occur at the plate end and increase with the radius of curvature. The limiting case of an infinite radius of curvature yields results identical to those given by Smith and Teng (2001).

In Figures 3 and 4, predictions from a finite element model represented by various symbols are also shown. In this finite element model, the beam and the plate were modelled using beam elements and connected by shear and tension springs (Teng and Zhang 2005). The close agreement between the analytical and finite element results seen in these figures demonstrates the correctness and accuracy of the analytical solution.

![Figure 3](image3.png)  
**Figure 3** Interfacial stresses in a CFRP-plated curved RC beam subjected to a mid-span point load

**CONCLUSIONS**

In this paper, a closed-form solution for the interfacial shear and normal stresses in curved beams of uniform section size under a point load strengthened with an externally bonded thin plate has been presented. The solution is based on some key simplifying assumptions: the interfacial stresses do not vary across the thickness of the adhesive layer, the effect of shear deformations is neglected, and the compatibility equations are written as for a straight beam assuming that the radius of curvature is large compared to the beam cross-sectional height. Predictions from this analytical solution have been shown to be in close agreement with appropriate finite element predictions, demonstrating the correctness and accuracy of the analytical solution. The main conclusion that can be drawn from the present study is that, in the elastic range, the concentration of interfacial shear and normal stresses at the plate end decreases with the radius of curvature. This implies that plate-end debonding is less critical for curved plated beams than it is for straight beams.

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