ANALYTICAL AND FINITE ELEMENT STUDY OF CRACKING AND INTERFACIAL STRESS DISTRIBUTION AT THE CRACKED ZONE IN A RETROFITTED RC BEAM

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ABSTRACT

Nowadays composite materials have a significant role in structural retrofitting and repairing. The failure of retrofitted members is usually due to delamination of the plate from the beam. This mode of failure may occur either at the plate end or at the cracked zone in a concrete beam. This paper aims to examine the critical interfacial shear stresses at the vicinity of the cracked region. The methodology consists of analytical and finite element solutions. A closed form solution is developed to express the distribution of interfacial shear stresses along the bond length at the cracked zone of a reinforced concrete beam. In the analytical solution the nonlinear behavior of concrete is examined. In this part, by solving equilibrium equations based on the strain compatibility condition, the tensile force in FRP fabric would be calculated as a boundary condition in defining the interfacial shear stress function. Also, the numerical solution using FEM is performed in order to control the analytical predictions. In this part the concrete is linear elastic and the pattern of a single crack formation at the midspan is prepared.

KEYWORDS

FRP fabrics, retrofitting, crack, interfacial stresses, bond length.

INTRODUCTION

Recently, using fiber reinforced plastics has been researched by the engineers, widely. Considering their notable specifications such as increasing the load caring capacity of structures, high tensile strength, long term durability, corrosion and low weight, composite materials have almost replaced steel plates as externally reinforcement for concrete members.

Several modes of failure such as: FRP rupture, concrete crushing, compressive failure, and debonding, have been discussed by researchers. These failure modes are discussed in details in (Teng et al. 2002a). The debonding failure mode which occurs between FRP sheet and concrete substrate is a common structural failure mode. Thus, recently the bond behavior has been discussed and analyzed by the researchers, widely. The researches referred to (Teng et al. 2002b) and (Pesic et al. 2002) presented stress variation plots of the FRP-anchorage zone based on the linear FE analysis. The studies related to (Taljsten 1997) and (Malik et al. 1998), developed analytical solutions for the shear and normal stresses along the interface between the bonded plate and the concrete beam.

The work performed by (Smith et al. 2001) prepared linear analytical models and compared them with other solutions. In a similar work, some parametric solutions were made and compared to the FE solution (Lau et al. 2001). Also, from the finite element solution approaches for estimating the stress variations at the plate end the work done by (Yang et al. 2004) is considerable.

Although the interfacial stresses at the plate end play a significant role in the debonding failure mode of the FRP retrofitted concrete beams, the interfacial stresses at the cracked zone are more critical. In concrete beams, flexural and shear cracks are commonly found on the tensile side of the beam. Under loading, these cracks tend to open and induce high interfacial shear stresses. A recent work made a fracture mechanics based analysis to obtain the maximum interfacial shear stress at the cracked region, for a given applied moment (Leung et al. 2001).
OBJECTIVE OF THE CURRENT STUDY

The main object of the present paper is to perform analytical and FE solutions for defining the interfacial shear stresses in the adhesive layer between FRP and concrete beam, at the cracked region of the beam, and compare them with each other. In the analytical study, a solution is applied for calculation of tensile force in the FRP fabric as a boundary condition to predict the shear stress function at the cracked zone, where the concrete has nonlinear behavior. This method performs equilibrium equations based on strain compatibility condition.

In order to verify the mathematical results, numerical finite element solution is presented. In this part of the study, a crack pattern is simulated in which for estimating the height of the crack, the fracture mechanics based equations are employed.

METHOD OF SOLUTION

Interfacial Shear Stresses at the Plate End

In order to study stresses at the cracked region, one should first generate the shear stress relation in the adhesive layer at the plate end. First, it should be noted that in this solution, the adhesive behaves elastically and linearly. Also, the stresses in the adhesive layer do not change with thickness. The transformed moment of inertia and area of the section denoting by \( I_{trc} \) and \( A_{trc} \), are calculated for the modeled beam including concrete and tensile steel reinforcement. Figure 1 shows the loading condition and the beam section.

Considering \( \varepsilon_p(x) \) and \( \varepsilon_c(x) \) as strains in FRP and concrete layers respectively, the differentiation of the shear stress with respect to \( x \), is:

\[
\frac{d\tau(x)}{dx} = \frac{G_a}{t_a} \left( \varepsilon_p(x) - \varepsilon_c(x) \right) \tag{1}
\]

Where \( G_a \) and \( t_a \) are the shear modulus and the thickness of the adhesive layer, respectively. The mechanical properties for the FRP plate behave isotropically and homogeneously. Moreover, the laminate theory is used to estimate the strain of the composite plate (Hyer 1998). Assume that the ply arrangement of the plate is symmetrical with respect to the mid-plane axis of the FRP plate, a grate simplification in laminate analysis then occurs by assuming that the extensional-bending coupled matrix is identically zero. Furthermore, it is assumed that no external bending moment is applied to the plate (Lau et al. 2001). Thus, the strain relation for the plate would be:

\[
\varepsilon_p(x) = \frac{N(x)}{E_p t_p w_p} \tag{2}
\]

Where \( E_p \) is the FRP effective modulus of elasticity, \( t_p \) and \( w_p \) are the FRP plate thickness and width respectively. Also, the strain in the concrete is:

\[
\varepsilon_c(x) = \frac{M_c(x) y_C}{E_c I_{trc}} - \frac{N(x)}{E_c A_{trc}} \tag{3}
\]

Where \( M_c(x) \) is the applied moment on the concrete layer, \( N(x) \) is the axial force acting on the concrete or the FRP layer, \( E_c \) and \( y_c \) are the concrete modulus of elasticity and distance from the bottom of the concrete layer to its centroid, respectively.

From the section equilibrium, one can write the relation between the total moment in the section and the moment applied in the concrete layer \( M_c(x) \), and the moment applied in the FRP layer \( M_p(x) \) (Smith et al. 2001):

\[
M_T(x) = M_c(x) + M_p(x) + N(x) \left( y_c + y_p + t_a \right), \quad M_c(x) = R M_p(x), \quad R = \frac{E_c I_{trc}}{E_p t_p} \tag{4}
\]

\( y_p \) is the distance from the top of the FRP layer to its centroid and \( I_p \) is the moment of inertia of the FRP plate. By substitution of (2) to (4) into (1) and solving the resulted differential equation with its boundary conditions, the shear stress relation becomes:
\[ \tau(x) = C_1 \cosh(\beta x) + C_2 \sinh(\beta x) + \eta_0 \]  

(5)

\[ C_1 = -C_2 = \frac{G_a y_c R_b d}{t_a E_c t_{trc}} , \quad \eta_0 = \frac{R G_a y_c R_b}{(R + 1) E_c t_{trc}} \beta^2 , \quad \beta_0 = \left[ \left( \frac{1}{E_p t_p} + \frac{R}{R + 1} \frac{y_c}{E_c t_{trc}} - \frac{w_p}{E_c A_{trc}} \right)^{0.5} \right] \]

(6)

Interfacial Shear Stress at the Cracked Location at Midspan of the Beam Using Strain Compatibility Condition

Using strain compatibility, the equilibrium equations are performed at the cracked section of the beam. At this section, concrete has nonlinear behavior and the tensile part is neglected.

Considering the nonlinear behavior of the concrete, geometry of the section and strain compatibility as shown in Figure 2, the equilibrium equations at the cracked section are:

\[
\begin{align*}
C_c - f_c A_p & = 0 \\
C_c \bar{y} + A_p f_c (d_p - c) & = M_f(x)
\end{align*}
\]

(7)

![Figure 2. Geometry of the section and strain compatibility](image)

Where \( C_c \) and \( \bar{y} \) are the overall compression force in the concrete under pressure and its distance from the neutral axis, respectively. The equilibrium equations mentioned will help to find the FRP tensile force at the cracked region.

At the cracked zone the tensile force in the FRP plate is \( N_p \). Calculation of \( N_p \) via the equilibrium equations mentioned, would result in a boundary condition which helps to find the shear stress relation. In order to obtain the shear stress relation at the cracked region, the variable \( x \) should be transformed into \( \frac{L}{2} - x \) in (5) as the following:

\[ \tau(x) = C_1 \cosh \left( \beta \left( \frac{L}{2} - x \right) \right) + C_2 \sinh \left( \beta \left( \frac{L}{2} - x \right) \right) + \eta_0 \]

(8)

Finite Element Solution

For the validation of the analytical results, a finite element approach is performed in this study. As illustrated in Figure 3, a two dimensional model is adopted for the numerical analysis. For concrete, adhesive and FRP used in this model, 8-node quadrilateral element is chosen. In this solution, the mechanical specifications of the adhesive itself are identical in all directions, i.e., it has an isotropic behavior. For the crack tip, 6-node triangle element is selected.

A Description on the Presented Model

The reinforced concrete beam is modeled as a two dimensional plane stress beam. Using the benefit of symmetry, only half of the beam is modeled. The model is cut along the longitudinal and transverse axes of the beam. For examining the stress distribution, the meshing pattern is fined at the cracked location at the midspan of the beam as shown in Figure 3.

The element used for modeling of concrete, adhesive and FRP layers is 2-D 8-Node Structural Solid. According to the fracture mechanics based theory; the elements which are used at the crack tip are singular 6-node triangle elements. In these singular elements, the midside nodes are placed at the quarter points. The meshing pattern at the crack tip is shown in Figure 4.
For reasonable results in modeling the crack tip, the first row of elements around the crack tip should have a radius of approximately a/8 or smaller, where a is the crack height. In the circumferential direction, roughly one element every 30° or 40° is recommended.

An important feature which must be considered for the crack pattern modeling is the crack height. In this paper, stress in the concrete at the upper point of the crack tip, is a criteria for estimating the crack height. In this procedure, for a definite point load at the crack location, a trial and error process is applied to define the crack height. For some alternative crack heights the meshing at the crack tip is prepared until the stress at the upper point of the crack tip would be smaller than the tensile strength of the concrete (in this research 4 MPa).

RESULTS AND DISCUSSIONS

Results Verification

Based on the analytical and FE solutions, the results are presented in this section. The geometry and loading condition of the model is a beam under three-point bending as illustrated in Figure 1. The geometry and materials specifications used for the verification are: \( L_p=2000\text{mm}, \ L=1880\text{mm}, \ \omega=150\text{mm}, \ h=200\text{mm}, \ E_p=235\text{GPa}, \ E_a=0.992\text{GPa} \). The compressive strength of the concrete is \( f'_c=42.9\text{MPa} \) and the yield stress of the steel reinforcement is \( f_y=200\text{MPa} \). The beam is under a point load of 40 kN at the mid span. In this research, a parametric study is prepared for observing the influence of tensile steel reinforcement, FRP layer thickness and adhesive layer thickness on the shear stress variation at the cracked zone at the mid span of the beam. Thus, the steel reinforcement \( A_s=2\Phi 10\text{mm} \) and \( A_s=2\Phi 20\text{mm} \) are examined. Also, FRP layer thicknesses \( t_p=1,1.5,2 \text{mm} \) and adhesive layer thicknesses \( t_a=1,1.5,2 \text{mm} \) are simulated for the model.

For simulating the crack pattern, it is notable that in the FE model, the crack penetrates into the adhesive layer (Leung et al. 2001). As shown in Figure 5, if the crack stays at closed at the interface between the concrete and the adhesive, a stress singularity will exist at the tip. Since the crack is going from a stiffer material (concrete) into a more flexible one (adhesive), this singularity is magnified (Bogy 1971). Moreover, most common adhesives are quite brittle and therefore it is highly likely for the crack to penetrate the adhesive, as illustrated in Figure 5. Also, as previously depicted, at the crack tip singular elements are used which are shown in Figure 4.

The results presented here introduce parametric studies to compare the stress at the cracked zone obtained from the FE and the analytical solutions. For the whole parametric studies, values of the crack height are estimated as: \( a=80 \text{ mm}, \ a=100 \text{ mm} \) and \( a=110 \text{ mm} \). These meshing patterns are illustrated in Figures 6-8.

There is an important feature about the stress variations at the vicinity of the cracked zone of the modeled beam, That is, the length along which the stress dissipates from its peak point at the cracked location at the mid span to the minimum which is ignorable. This length, named in this work as the dissipating length, is illustrated geometrically on the stress variations and compared for the FE and the analytical solutions.
The first part of the results section is allocated to the examining of the effect of the tensile steel reinforcement on the shear stress variation at the cracked zone. In this part, the thicknesses of the adhesive and the FRP layers are: \( t_a = 1 \text{mm} \) and \( t_p = 2 \text{mm} \). The shear stress variations in the adhesive layer at the vicinity of the cracked zone at the mid span of the modeled beam, obtained from the FE and the analytical solutions are presented in Figures 9 and 10. These results indicate that, as the steel reinforcement increases, the stress in the adhesive layer and also the crack height decrease. This decrease is due to the fact that increasing the tensile steel reinforcement results in reduction of the movement of the crack faces and consequently, the decrease in the stress in the adhesive layer and the crack height would be unavoidable.

Another important feature which must be considered is the boundary length in which the interlaminar shear stress is unignorable and dissipates from its maximum value to an ignorable one. This length is introduced here as the dissipating length and is illustrated in Figures 9 and 10 as \( l_d \). This length is plotted geometrically on the stress variations and compared for the FE and the analytical solutions. At a definite distance from the cracked region, the stress variations asymptote to a very small value which can be obtained geometrically by drawing a horizontal tangent line to the variations, as illustrated in Figures 9 and 10 with a dashed line. The intersection of this line and the variations would be a point in which the stress is negligible and almost constant from the variations asymptote point up to the zone at the vicinity of the cut off point (that is the plate end) near the supports. Consequently, the dissipating length would be defined as the horizontal distance between the variations asymptote point and the location of the peak stress at the mid span.

A results summary is presented in table 1. Due to the FE and the analytical solutions in the table below, tensile strain in the FRP layer, \( \varepsilon_{PT} \), and stress in the tensile steel reinforcement, \( f_s \), at the cracked zone at the mid span are depicted.
Table 1. Summary results for the parametric study on tensile steel reinforcement ($ta=1mm$, $tp=2mm$)

<table>
<thead>
<tr>
<th>$A_s$</th>
<th>$\varepsilon_{PT}$</th>
<th>$f_s$(MPa)</th>
<th>$\varepsilon_{PT}$</th>
<th>$f_s$(MPa)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>FE Analytical</td>
<td>FE Analytical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.001351 0.001579</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$2\Phi 10mm$</td>
<td>0.001136 0.001304</td>
<td>200</td>
<td>175.92</td>
<td></td>
</tr>
<tr>
<td>$2\Phi 20mm$</td>
<td>0.0007532 0.00106</td>
<td>119.73</td>
<td>134</td>
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</table>

From the table it can be observed that by increasing the steel reinforcement, the strain in the tensile FRP and the stress in the steel reinforcement decrease.

In the next section of the results, effect of the thickness of the FRP layer on the stress distribution at the cracked zone, is examined. The results are illustrated in Figures 11 and 12. Here tensile steel reinforcement $A_s=2\Phi 10 mm$ and the thickness of the adhesive layer $ta=1mm$ are assumed and the shear stress distribution at the vicinity of the cracked zone at the mid span is plotted. These simulations indicate that as the thickness of the FRP layer increases, the stress in the adhesive layer and also the crack height decrease. This would be because of the fact that the FRP layer works as a reinforcement for the structure and would not let the faces of the crack move easily. Thus, this procedure would result in decrease in the stress distribution in the interface between the concrete and the FRP layer. Again, the dissipating lengths are illustrated in Figures 11 and 12 and compared for the FE and the analytical solutions.

A results summary for this section is presented in table 2. Due to the FE and the analytical solutions in the table below, tensile strain in the FRP layer and stress in the tensile steel reinforcement at the cracked zone at the mid span are presented regarding the variety of the thickness of the FRP layer.

From the table it can be observed that by increasing the FRP layer thickness, the strain in the tensile FRP layer and also the stress in the steel reinforcement decrease.

Table 2. Summary results for the parametric study on FRP layer thickness ($A_s=2\Phi 10 mm$, $ta=1mm$)

<table>
<thead>
<tr>
<th>$tp$ (mm)</th>
<th>$\varepsilon_{PT}$</th>
<th>$f_s$(MPa)</th>
<th>$\varepsilon_{PT}$</th>
<th>$f_s$(MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FE Analytical</td>
<td>FE Analytical</td>
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<td>1.5</td>
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<td>200</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.001136 0.0013039</td>
<td>119.73</td>
<td>134</td>
<td></td>
</tr>
</tbody>
</table>

Finally, for the last section of the results, effect of the thickness of the adhesive layer on the stress variations at the cracked location is examined. The results are illustrated in Figures 13 and 14. The tensile steel reinforcement $A_s=2\Phi 10 mm$ and the thickness of the FRP layer $tp=2mm$ are assumed and the shear stress distribution at the vicinity of the cracked zone at the mid span is plotted. These simulations indicate that as the thickness of the adhesive layer increases, the stress in the adhesive layer decreases but the crack height increases. This is because of the fact that as...
the thickness of the adhesive layer increases, the attachment of the FRP layer to the concrete increases which would result in the reduction of the stress distribution at the cracked zone. On the contrary, since the adhesive works in shear it can not dominate the tensile forces applied at the cracked location. Thus, this effect at the cracked zone would be a slightly increase in the crack height.

The dissipating lengths are also illustrated in Figures 13 and 14 and compared for the FE and analytical solutions.

From the Figures 13 and 14 it can be seen that the dissipating length obtained from the two methods are almost identical.

At last, for this section a results summary is depicted in table 3. Due to the FE and analytical solutions in the table below, tensile strain in the FRP layer and stress in the tensile steel reinforcement at the cracked zone at the mid span are presented regarding the variety of the thickness of the adhesive layer.

From the table 3 it can be observed that by increasing the adhesive layer thickness, strain in the tensile FRP layer decreases, however, stress in the steel reinforcement remains almost constant but tends to decrease.

**Table 3. Summary results for the parametric study on adhesive layer thickness (A_s=2\(\Phi_{10}\)mm, t_P=2mm)**

<table>
<thead>
<tr>
<th>t_a (mm)</th>
<th>(\varepsilon_{PT})</th>
<th>(f_s) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FE</td>
<td>Analytical</td>
</tr>
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<tr>
<td>2</td>
<td>0.001086</td>
<td>0.001299</td>
</tr>
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</table>

**CONCLUSIONS**

Based on the current research, the interfacial stress distribution in a retrofitted RC beam is seen to be critical at the cracked zone. This is mostly because of the effect of the crack opening on the interfacial stresses in the adhesive layer. Analytical and FE solutions are presented to predict these stresses at the cracked section when there is one single crack at the middle section of the beam. In order to define the stress distribution, the equilibrium method is implemented. For the numerical solution, singular elements are used for meshing pattern at the crack tip. Extended parametric studies are carried out to examine the influence of tensile steel reinforcement, FRP thickness and adhesive thickness on the interfacial shear stress distribution.

It can be concluded that considering the good agreement between the analytical and finite element solutions, the analytical solution here using strain compatibility condition, would be a reasonably substitution for the prediction of the interfacial stresses at the cracked location.

Another feature discussed in this paper, is the introduction of the dissipating length. As mentioned previously, along this length the peak stress at the cracked zone at the mid span of the beam reduces to a negligible value. This distance
is vital because the debonding of the FRP plate which is one of the most important failure modes in the retrofitted RC beams, takes place at this distance. In this research the dissipating length is depicted on the stress variations resulted from the FE and the analytical solutions. Considering the parametric studies discussed, it can be seen that the dissipating lengths plotted on the stress variations from the FE and the analytical solutions have a slight difference or they are almost identical.

REFERENCES